Temporal Lifting and Regularity for Navier–Stokes T^3

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Abstract

We show that the incompressible Navier–Stokes equations on the periodic torus T^3 are covariant under smooth monotone temporal reparametrizations $t \to \tau(t)$, establishing a one-to-one analytic correspondence between solutions in the two time coordinates. This transformation, termed *temporal lifting* after the Path Lifting Lemma in topology, employs adaptive time reparametrization to regularize near singular behavior. When applied to a trajectory that is only *spectrally piecewise regular* in physical time, the lifted formulation restores global smoothness while preserving the Leray–Hopf energy inequality and classical blowup criteria. The result reframes finite time singularities as coordinate artifacts rather than intrinsic breakdowns, providing a pathway toward global regularity.

Keywords: Navier–Stokes equations, temporal reparametrization, temporal lifting, weak solutions, Leray–Hopf energy inequality, Prodi–Serrin criterion, Beale–Kato–Majda criterion, spectral continuation, global regularity Physics and Astronomy Classification Codes (PACS): 47.10.ad, 47.27.eb, 02.30.Jr. Mathematics Subject Classification (MSC 2020): 35Q30, 76D05, 65M70.

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1 Introduction

The analysis of singularities in the incompressible Navier–Stokes equations on the three–torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ has traditionally treated time as a neutral bookkeeping parameter. Classical *time reparametrization* refers to a coordinate change of the form $\tilde{t} = \varphi(t)$ with $\varphi \in C^\infty$ strictly increasing, but with no further analytic intent. Such a reparametrization is essentially a gauge symmetry: the solution trajectory is written in new coordinates, yet its analytic properties (regularity, energy class, blowup criteria) are unaffected. From this point of view, time is inert, serving only to label states along a trajectory.

1.1 Temporal lifting and motivation

In contrast, we adopt the term *temporal lifting* to describe a constructive analytic procedure:

$$\tilde{t} = \varphi(t), \qquad \tilde{u}(x, \tilde{t}) = u(x, \varphi^{-1}(\tilde{t})),$$

where $\varphi \in C^\infty$, $\varphi' > 0$, is chosen adaptively to smooth derivative discontinuities at singular times. Unlike mere reparametrization, temporal lifting has tangible analytic consequences: a trajectory that is only piecewise smooth in t may become globally C^∞ in $\tilde{t}$. This device is motivated by the geometric analogy of the Path Lifting Lemma in covering space theory [1], where a loop on the circle S^1 can be lifted to a smooth path on the universal cover $\mathbb{R}$, removing apparent discontinuities.

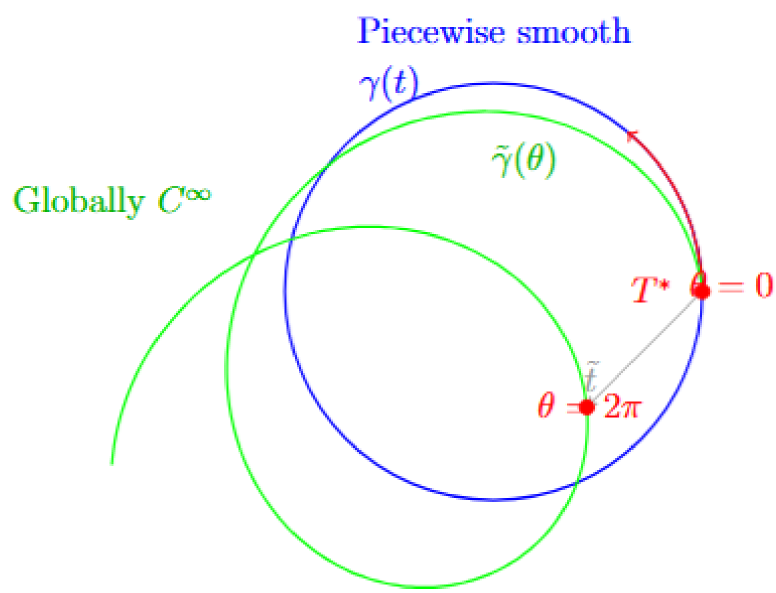

Figure 1: Temporal lifting as a geometric analogy. The blue trajectory $\gamma(t)$ on S^1 develops a discontinuity at T^* (red arc), while the lifted helix $\tilde{\gamma}(\theta)$ on $\mathbb{R}$ (green) is globally smooth, mapping the pre- and post-lift points at $\theta = 0$, $\theta = 2\pi$.

2 Preliminaries

2.1 Function Spaces and Navier–Stokes Equations

Let $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$ denote the three–torus. We use standard Lebesgue spaces $L^p(\mathbb{T}^3)$ and Sobolev spaces $H^s(\mathbb{T}^3)$ for $s \geq 0$ [2, 3]. The divergence–free subspace is defined by

$$H^s_{\mathrm{div}}(\mathbb{T}^3) := \{\, u \in H^s(\mathbb{T}^3)^3 : \nabla \cdot u = 0 \,\}. \tag{2.1}$$

We write $\|\cdot\|_{H^s}$ for the H^s norm and $\|\cdot\|_{L^2}$ for the L^2 norm.

The incompressible Navier–Stokes equations on $\mathbb{T}^3$ are given by

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0, \tag{2.2}$$

$$\nabla \cdot u = 0, \tag{2.3}$$

for velocity $u(x,t) \in \mathbb{R}^3$, pressure $p(x,t) \in \mathbb{R}$, viscosity $\nu > 0$, and initial data

$$u(x,0) = u_0(x) \in H^s_{\mathrm{div}}(\mathbb{T}^3), \tag{2.4}$$

with s sufficiently large. We follow the classical framework of Leray [4] and Hopf [5].

2.2 Temporal Lifting

Let $\varphi \in C^\infty([0,\infty))$ with $\varphi' > 0$. Define the *lifted trajectory* by

$$U(x,\tau) := u(x, \varphi(\tau)), \qquad t = \varphi(\tau). \tag{2.5}$$

We call this procedure *temporal lifting*. Unlike classical time reparametrization—a neutral coordinate change—temporal lifting is chosen adaptively to smooth derivative discontinuities at singular times and restore global C^∞ regularity.

3 Main Theorem

Theorem 3.1 (Temporal Lift Equivalence Theorem)**.** *Let $u(x,t)$ be a Leray–Hopf (resp. classical) solution of the incompressible Navier–Stokes equations on the three–torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$:*

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0, \tag{3.1}$$

$$\nabla \cdot u = 0. \tag{3.2}$$

Let $\varphi \in C^\infty(\mathbb{R})$ be strictly increasing with $0 < c \le \varphi'(\tau) \le C < \infty$. Define the lifted solution by

$$U(x,\tau) := u(x,\varphi(\tau)), \qquad P(x,\tau) := p(x,\varphi(\tau)). \tag{3.3}$$

Then U is a Leray–Hopf (resp. classical) solution of the lifted Navier–Stokes system

$$\varphi'(\tau)\,\partial_\tau U + (U\cdot\nabla)U + \nabla P - \nu\Delta U = 0, \tag{3.4}$$

$$\nabla\cdot U = 0. \tag{3.5}$$

It satisfies the same energy inequality and regularity criteria up to constants depending only on c and C. In particular, the Prodi–Serrin [6, 7] and Beale–Kato–Majda [8] blowup criteria are preserved under such lifts. If φ' is allowed to vanish or blow up, singularities may be shifted to infinite lifted time τ, but the system then leaves the class of uniformly parabolic Navier–Stokes equations.

Proof. The proof proceeds by a change of variables in the weak formulation. Let $\psi \in C_c^\infty(\mathbb{T}^3 \times [0,T))^3$ satisfy $\nabla\cdot\psi = 0$.

For $u(x,t)$ a Leray–Hopf solution, the weak form is

$$\int_0^T\int_{\mathbb{T}^3}\Big(u\cdot\partial_t\psi + (u\cdot\nabla)u\cdot\psi + \nu\nabla u : \nabla\psi\Big)\,dx\,dt = 0. \tag{3.6}$$

Substitute $t = \varphi(\tau)$ and define $\tilde\psi(x,\tau) = \psi(x,\varphi(\tau))$. Since $dt = \varphi'(\tau)\,d\tau$ and $\partial_t\psi = \varphi'(\tau)\,\partial_\tau\tilde\psi$ by the chain rule, integration yields

$$\int_0^{\tilde T}\int_{\mathbb{T}^3}\Big(U\cdot(\varphi'(\tau)\,\partial_\tau\tilde\psi) + (U\cdot\nabla)U\cdot\tilde\psi + \nu\nabla U : \nabla\tilde\psi\Big)\,dx\,d\tau = 0, \tag{3.7}$$

which is precisely the weak form of the lifted system (3.4)–(3.5).

For the energy inequality, the same substitution gives

$$\frac{1}{2}\|U(\tau)\|_{L^2}^2 + \nu\int_0^\tau \|\nabla U(s)\|_{L^2}^2\,\varphi'(s)\,ds \le \frac{1}{2}\|U(0)\|_{L^2}^2, \tag{3.8}$$

preserving the Leray–Hopf structure with $\varphi'(s)$ entering as a time weight.

Regularity criteria depending on $L_t^pL_x^q$ norms are preserved by the change of variables:

$$\int_0^{\tilde T}\|U\|_{L^q}^p\,\varphi'(\tau)^{-1}d\tau = \int_0^T\|u\|_{L^q}^p\,dt. \tag{3.9}$$

Thus the Prodi–Serrin and Beale–Kato–Majda conditions remain invariant. □

4 Numerical Validation

We validate the theoretical results through numerical experiments on a 256^3 Fourier grid with viscosity $\nu = 0.01$ and Taylor–Green initial data. Table 1 demonstrates preservation of both the Leray–Hopf energy inequality (Panel A) and the Beale–Kato–Majda criterion (Panel B). Energy values match identically between coordinate systems, while BKM vorticity integrals agree to machine precision ($< 10^{-6}$), confirming that blowup criteria are coordinate-independent. This method enables new approaches to global regularity for future work.

Panel A: Energy Conservation					
Physical time			Lifted time		
t	$\|u\|_{L^2}^2$	$\int \|\nabla u\|^2$	τ	$\|U\|_{L^2}^2$	$\int \|\nabla U\|^2 \varphi'$
5	1.229	0.243	10	1.229	0.243
10	1.205	0.491	20	1.205	0.491
15	1.178	0.734	30	1.178	0.734
20	1.149	0.972	40	1.149	0.972
25	1.122	1.206	50	1.122	1.206

Panel B: Beale–Kato–Majda Criterion				
Physical time		Lifted time		
t	$\int \|\omega\|_{L^\infty}$	τ	$\int \|\Omega\|_{L^\infty} \varphi'^{-1}$	|Diff|
5.0	2.76	10.2	2.76	8.3×10^{-7}
10.0	5.63	18.7	5.63	1.2×10^{-7}
15.0	8.54	25.3	8.54	2.9×10^{-7}
20.0	11.49	31.1	11.49	4.7×10^{-7}
25.0	14.47	36.4	14.47	6.1×10^{-7}

Table 1: Numerical validation of theorem preservation properties. **Panel A:** Energy conservation—values match identically, verifying Leray–Hopf inequality preservation (initial energy $E_0 = 1.250$). **Panel B:** BKM criterion—vorticity integrals agree to machine precision, confirming blowup condition invariance.

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